# Théorie des possibilités et modèles numériques linéaires pour l'inférence sur une base de croyances

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## Résumé

*En combinant logique possibiliste et modèles numériques linéaires, nous proposons un nouveau moyen d'inférer sur une base de croyances. Via l'introduction de degrés d'incertitudes, nous pouvons inférer malgré la présence d'inconsistence dans les informations fournies, et éventuellement les remettre en cause. Nous montrons que mêler modèles numériques et symboliques permet des inférences syntaxiques efficaces. Le tout est illustré à travers un exemple de l'aide à la décision multi-critères.*

## Mots-clés

*Logique possibiliste, Raisonnement sur base de croyances, Aide à la décision multi-critères, complexité, sémantique et syntaxe*

# 1 Introduction

In this paper, we propose a new framework for reasoning on a belief base. We use the term belief as the statements forming our base are possibly uncertain. In multiple fields, as in Multi-Criteria Decision Aiding (MCDA), it is common to reason over statements from the Decision Maker (DM), that are often uncertain. The purpose of the process is to help her[1] choose an alternative among several, each defined on a set of criteria. Her preferences are formalized as statements, from which it is possible to infer. With the hypotheses that her preferences can be modelled by a numerical model, each statement is a constraint that reduces the set of possible models. For a review of the numerical value models used to represent preferences, see [Bourdache, 2020]. Many frameworks for MCDA have been developed over the years. Classical DM preferential information consists of pairwise comparisons of alternatives, the pairs are usually chosen through some heuristics with the idea of improving the model fitness [Ciomek et al., 2017]. Translating this statement into constraints is usually done by excluding all models that do not satisfy the new constraint. An example of this kind of process can be found in [Jacquet-Lagreze and Siskos, 1982], where the UTA framework is introduced. In [Greco et al., 2008], it is extended to a robust framework. While *pointwise* models aim at finding a single value of the parameter of the model, which is usually done by minimizing some loss function, *robust* MCDA frameworks can make an inference only if all of the totally possible models support the inference. This differs from non-robust models, where procedures are applied to choose one of the models and use it to infer. Unfortunately, if the statements received are contradictory, inconsistency can appear during the process. In other words, there are no totally possible models. It may be caused by an error from the DM, or an error when the model was selected by the analyst. If all statements are considered certain, the situation is tricky, as a choice has to be made to continue the process. Previous work tends to solve the problem by removing formulas that cause inconsistency, as in [Mousseau et al., 2003, Greco et al., 2008]. However, if all statements are certain, removing one seems counter-intuitive.

To address this issue, [Greco et al., 2008] proposes as an extension to associate a degree of credibility with each piece of information contained in the base. This idea has been developed in [Adam and Destercke, 2024], where belief bases are combined with reasoning under inconsistency. The paper proposes to represent beliefs within the scope of possibility theory. Possibility theory has been developed thanks to the preliminary work of Zadeh in [Zadeh, 1978]. It is built upon fuzzy sets, sets where each element has an ordinal priority degree associated. Within possibility theory, the degree of priority is the degree of credibility of each statement. For a complete overview of the possibility theory, we invite the reader to refer to [Dubois and Prade, 2015, Dubois and Prade, 2004].

As well as providing a representation of uncertainty, possibility theory is of interest when it comes to reasoning as it introduces logic tools to infer over a set of beliefs stratified by credibility, as in [Dubois et al., 1994]. As each statement is uncertain at a given degree, it is possible to attach a certainty degree to pieces of information deducted from the base, depending on which statement allow the deduction. This reinforces the trust the analyst has in the presented deductions. Furthermore, tools for reasoning

1. We use the neutral feminine for the DM

under inconsistency in possibilistic logic are developed in [Benferhat et al., 1999]. These tools have not been used in [Adam and Destercke, 2024] where they introduce numerical ones for reasoning. We believe that the semantics for reasoning under inconsistency from possibilistic logic are interesting, as they permit the definition of computationally good syntaxes providing certificates for the analyst. Hence, the aim of this article is to use the possibilistic reasoning framework with linear numerical models. The benefits of this combination are the appearance of the degree of credibility and the tools for reasoning provided by the logical framework.

A research gap also exists within possibility theory, where little attention was given to computational results. [Lang, 2000] proposes algorithms and computational results when the beliefs are expressed using propositional logic. There is a need to define syntaxes for possibility theory applied to beliefs expressed as numerical formulae and give computational results. These syntaxes should be created while keeping in mind the explanation, as in [Belahcene et al., 2017, Amoussou et al., 2023], process that plays a significant part in the trust that our analyst will place in our decisions. Even though this is not the current scope of our work, this is a natural extension.

Section 2 will present possibility theory and the semantics of possibilistic logic, as seen in the literature. In Section 3, we restrict our universe to beliefs expressed as linear inequalities. We develop new syntaxes for inferring and give computational results. In Section 4, we give a possible application of our logic of linear comparison, MCDA. Our proposal is discussed regarding the literature in Section 5. We end the paper with some conclusions in Section 6.

## 2 Possibilistic logic : semantics

Let us have $\Omega$, the set of states of the world, with $\omega \in \Omega$, a state. A possibility distribution $\pi$ is a mapping from $\Omega$ to $[0, 1]$. It represents the plausibility of each state of the world, the higher, the more plausible. $\pi(\omega) = 1$ means that the state is totally possible, $\pi(\omega) = 0$ that it is impossible.

A belief base is built on a set of formulae, each associated with a level of belief. We assume formulae belong to a set $\mathcal{F}$ called the *hypotheses class*, e.g. propositional formulae over a given language, or linear comparisons between numerical values. For $\phi \in \mathcal{F}$, we note $[\![\phi]\!] \subseteq \Omega$ the subset of $\omega$ that satisfies $\phi$. We associate a value $\alpha$ with $\phi$, taken in an ordinal set. It represents the priority of its associated formula, its credence, compared to the other formulae present in the possibilistic base $\Gamma$.
The degree of belief possess a clear semantics. It can be linked to imprecise probabilities as defined by [Walley, 1991, Dubois et al., 1993]. Given a formula $\phi$, consider the lottery in which a gambler receives a unit gain if $\phi$ is satisfied and zero otherwise. $\alpha$ is defined as the maximum purchase price one would pay to enter this game. In particular, $\alpha$ is one iff the gambler is certain of a positive outcome, otherwise she would forfeit a sure gain or incur a sure loss.
The statement $(\phi, \alpha)$ claims the worlds where $\phi$ is satisfied are fully possible and the world where $\phi$ is not are partially possible, at level $1 - \alpha$, as captured by Equation (1).

$$\pi_{(\phi_j,\alpha_j)}(\omega) = \begin{cases} 1 & \text{if } \omega \in [\![\phi_j]\!], \\ 1 - \alpha_j & \text{otherwise.} \end{cases} \tag{1}$$

When dealing with a conjunction of statements, we opt for the *minimal specificity* criterion where the possibility level is aggregated using the minimum operator. This allows to define the possibility distribution induced by a belief base $\Gamma$ according to Equation (2).

$$\pi_\Gamma(\omega) = \min_{(\phi_j,\alpha_j)\in\Gamma} \pi_{(\phi_j,\alpha_j)}(\omega) \tag{2}$$

This definition extends the case where information is perfect : when levels are restricted to $\{0, 1\}$, a new statement $\phi$ prunes the set of possible world by asserting all worlds that do not satisfy $\phi$ are completely impossible. Observe that, when $\alpha_1 > \alpha_2$, the statement $(\phi, \alpha_1)$ subsumes the statement $(\phi, \alpha_2)$.

**Example 1.** *We have a bag. Inside the bag there could be nothing, a banana, an apple, or both. We have* $\Omega = \{ba, \bar{b}a, b\bar{a}, \bar{b}\bar{a}\}$. *We receive the information that* $(a \oplus b)$ *with a certainty of* 0.9. *Hence,* $\Gamma^1 = \{(a \oplus b, 0.9)\}$. *In other words, the necessity that there is exactly a fruit in the bag is superior or equal to* 0.9. $\Gamma^1$ *induces a new possibility distribution for all states of the world, as shown in Table 1.*

| $\omega$ | $\pi_{\Gamma^1}$ |
|---|---|
| $ba$ | 0.1 |
| $\bar{b}a$ | 1 |
| $b\bar{a}$ | 1 |
| $\bar{b}\bar{a}$ | 0.1 |

TABLE 1 – Possibility distribution induced by $\Gamma^1$

*The states of the world not satisfying the formula in* $\Gamma^1$ *become less possible, as in Equation (1). However, states of the world satisfying the piece of information are still totally possible. If we consider* $\Gamma^2 = \{(a \oplus b, 0.9), (b, 0.7)\}$, *we have the possibility distribution of Table 2.*

| $\omega$ | $\pi_{\Gamma^2}$ |
|---|---|
| $ba$ | 0.1 |
| $\bar{b}a$ | 0.3 |
| $b\bar{a}$ | 1 |
| $\bar{b}\bar{a}$ | 0.1 |

TABLE 2 – Possibility distribution induced by $\Gamma^2$

*For* $\bar{b}\bar{a}$ *not satisfying both formulas, we take the minimum possibility value between* $1 - 0.9$ *and* $1 - 0.3$, *as shown in Equation (2).*

If an $\omega$ does not satisfy $\phi_j$, we give it a possibility value equal to $1 - \alpha_j$, as in Equation (1). Hence, if $\phi_j$ has a strong credence, an $\omega$ not satisfying it will be deemed less possible. To compute the possibility distribution for $\Gamma$, we simply take the minimum value returned by each pair, as in Equation (2). Therefore, if an $\omega$ belongs to all the $[\![\phi_j]\!]$, its possibility value will be 1, it is totally possible. However, if such an $\omega$ does not exist, we say that the base is inconsistent. The level of inconsistency is evaluated as in Equation (3).

$$Inc(\Gamma) = 1 - \max_{\omega \in \Omega} \pi_\Gamma(\omega) \tag{3}$$

If the base is consistent, $Inc(\Gamma)$ is equal to 0. Otherwise, it is at least of value $1 - \alpha_j$ where $\alpha_j$ is the lowest value among the pairs $(\phi_j, \alpha_j)$.

The handling of uncertainty in possibility theory sees three advantages. Firstly, it captures standard set theory, as a possibility is coherent whenever its maximum reaches one (therefore, multiple elements can have possibility one). This contrasts with probabilities, where the summation constraint means that making one element more plausible decreases the plausibility of some others, making it unable to capture set theory. Secondly, the coherence of the final result is not considered an axiom. This differs, e.g., from standard probabilities or even from lower previsions [Walley, 1991]. Thirdly, the user does not require to know how new numbers are computed, as the computations are done with an ordinal setting and the min operator.

Finally, we introduce the notion of $\alpha$-cuts. These are subbases $\Gamma_{>\alpha}$ and $\Gamma_{\geq\alpha}$, defined in Equation (4).

$$\begin{aligned} \Gamma_{>\alpha} &= \{(\phi_j, \alpha_j) \in \Gamma \text{ s.t } \alpha_j > \alpha\} \\ \Gamma_{\geq\alpha} &= \{(\phi_j, \alpha_j) \in \Gamma \text{ s.t } \alpha_j \geq \alpha\} \end{aligned} \tag{4}$$

We simply do not consider pairs with an $\alpha_j$ (strictly or not) below the given threshold. This will prove useful when dealing with inconsistent bases, and allows for a concise characterization of the inconsistency level :

$$Inc(\Gamma) = \alpha \iff \Gamma_{>\alpha} \text{ is consistent but not } \Gamma_{\geq\alpha} \tag{5}$$

**Example 2** (Example 1 continued)**.** *We now have* $\Gamma^3 = \{(a \oplus b, 0.9), (b, 0.7), (\bar{a} \wedge \bar{b}, 0.6)\}$. *Our analyst tells us that the necessity that there are no fruits inside the bag is at least* 0.6. *This makes the base inconsistent, and we have the possibility distribution of Table 3.*

| $\omega$ | $\pi_{\Gamma^3}$ |
|---|---|
| $ba$ | 0.1 |
| $\bar{b}a$ | 0.3 |
| $b\bar{a}$ | 0.4 |
| $\bar{b}\bar{a}$ | 0.1 |

TABLE 3 – Possibility distribution induced by $\Gamma^3$

*This base is inconsistent. If we solely consider the base without the degree of certainty, the analyst is affirming that there are fruits and there are no fruits in the bag. It is totally contradictory. Without degrees, to repair such an inconsistency, one would have to choose between either the first two pieces of information or the last. However, we have access to additional information with the degrees of certainty. A basic way to repair the belief base is to consider the formulae on the degree of inconsistency, as defined in Equation (3). As the most plausible world is $b\bar{a}$ with degree 0.4, we have* $Inc(\Gamma^3) = 0.6$. *In other words, the strict* 0.6*-cut of* $\Gamma$, $(\Gamma^3)_{>0.6}$, *is consistent. It amounts to ignoring formulae with level 0.6 or lower, which actually yields* $\Gamma^2$.

With the basics of possibility theory clearly defined, we can now focus on the possibilistic inference of a formula $\phi$ from a possibility distribution induced by $\Gamma$, $\pi_\Gamma$. When information is certain (i.e., all $\alpha$s are one), inferring on such a base is equivalent to *robust* (or *skeptical*) inference : checking that all the totally possible states of the world satisfy the formula we want to infer. The possibilistic inference semantics extends this process to imperfect information, with level in $[0, 1]$ and beliefs that are not necessarily consistent.

**Definition 1.** **([Dubois et al., 1994], Possibilistic inference)** : $\Gamma \models_{pi} (\phi, \alpha)$ iff $\forall \omega, \pi_\Gamma(\omega) \leq \pi_{\{(\phi,\alpha)\}}(\omega)$

**Example 3** (Example 1 continued)**.** *We consider* $\Gamma^2$. *It is possible to infer $\bar{a}$, i.e. the absence of an apple, from the base. The two possibilities distribution are in Table 4.*

| $\omega$ | $\pi_{\Gamma^2}$ | $\pi_{(\bar{a},\alpha)}$ |
|---|---|---|
| $ba$ | 0.1 | $1-\alpha$ |
| $\bar{b}a$ | 0.3 | $1-\alpha$ |
| $b\bar{a}$ | 1 | 1 |
| $\bar{b}\bar{a}$ | 0.1 | 1 |

TABLE 4 – Possibility distribution induced by $\Gamma^2$ and $\bar{a}$

*With* $\alpha = 0.7$*, for all possible states of the world, the possibility of having $\bar{a}$ is greater than the possibility of* $\Gamma^2$. *Hence, we can infer* $(\bar{a}, 0.7)$ *from* $\Gamma^2$. *In other words, from the facts we believe it is 0.9-certain there is one fruit in the bag and it is 0.7-certain there is a banana, we can deduce it is 0.7-certain there is no apple.*

Possibilistic inference is similar to what is done in other kinds of logic. For example, Definition 1 can be rewritten as a refutation. Instead of checking the inclusion of all totally possible $\omega$ in $[\![\phi]\!]$, it is equivalent to verify that the intersection between the totally possible $\omega$ and $[\![\neg\phi]\!]$ is empty. However, when the base is inconsistent ($Inc(\Gamma) > 0$), we can infer any formula at $\alpha = 1 - Inc(\Gamma)$ because the empty set is included in every set. At this point, it is necessary to remember the difference between inferring and deciding.

Inferring $\phi$ and $\neg\phi$ on an inconsistent base without degrees of certainty is problematic, as it does not come with tools helping to solve the conundrum. By contrast, the higher-order information provided by the valuation of beliefs with degrees of certainty can help to address the problem, as the decision has to be made between $(\phi, \alpha)$ and $(\neg\phi, \beta)$, a richer information. As decision is not the focus of this work, we leave this question open. Two new types of inference will be introduced, specially built to deal with the inconsistent case. We introduce the first semantics, the *non-trivial inference*, in Definition 2.

**Definition 2.** **([Dubois et al., 1994], Non-trivial inference)** $\Gamma \models_{nt} (\phi, \alpha)$ iff $\pi_{\Gamma_{>Inc(\Gamma)}}(\omega) \leq \pi_{\{(\phi,\alpha)\}}(\omega)$

Equivalently, this can be written :

$$\Gamma \models_{nt} (\phi, \alpha) \text{ iff } \Gamma_{\geq\alpha} \text{is consistent and } \Gamma_{\geq\alpha} \models_{pi} (\phi, \alpha) \tag{6}$$

The semantics is based on the possibility degrees. We simply ignore the formulas under the inconsistency level. As $\Gamma_{>Inc(\Gamma)}$ is always consistent, inferring is safe. Here, inference and decision are equivalent, as it is not possible to infer $\phi$ and $\neg\phi$ on a consistent base. However, the disadvantage of this semantics is the drowning effect : all information below the inconsistency level is lost, even though some of it might be consistent with the curated base. Hence, the accepted inferences will be high in terms of certainty, but less pieces of information will be inferred. For a specific discussion of this issue, refer to [Benferhat et al., 1999, Section 3.2]. To avoid this, we introduce a last semantics to tackle inconsistency, the safe possibilistic inference in Definition 3.

**Definition 3.** **([Benferhat et al., 1999], Safe inference)** $\Gamma \models_s (\phi, \alpha)$ iff $\exists \Gamma^* \subseteq \Gamma$ s.t $\max_\omega \pi_{\Gamma^*}(\omega) = 1$ and $\forall \omega, \pi_{\Gamma^*}(\omega) \leq \pi_{\{(\phi,\alpha)\}}(\omega)$

If a consistent subbase of $\Gamma$ supports the inference of $(\phi, \alpha)$, then this inference is deemed safe. In this case, inferring and deciding are different. Two different coherent subbases could allow to respectively infer $\phi$ and $\neg\phi$. In the literature, the quality of a subbase is measured by the minimum degree of certainty of one of its formulas. If we can infer $\phi$ and $\neg\phi$, deciding $\phi$ would require the best subbase for $\phi$ to have a strictly better rank than the best for $\neg\phi$. However, before turning to the decision mechanism, it is necessary to construct syntaxes for our inference process.

Now that a very brief definition of possibility theory and semantics for inference have been given, let us restrain our $\Omega$ so that we can give computational results and define a syntax for our semantics given in Definitions 1, 2 and 3.

# 3 The logic of linear comparison

## 3.1 Restricting to linear inequalities

Our aim is to use the expressivity and good computational properties of linear models while keeping the logical tools for reasoning over a base. To our knowledge, very few works in the field of possibility theory have covered this point. [Adam and Destercke, 2024] proposes to use possibility theory to address inconsistency in a MCDA setup, but does not use possibilistic tools to perform the inference.

From now on, we restrict $\Omega$ to the Cartesian product of domains defined over a continuous space. We have $\Omega = (\mathbb{R}_+)^n \setminus \{0\}$. Therefore, each $\omega \in \Omega$ is a non-null vector of $n$ dimensions in $\mathbb{R}_+$. Without loss of generality, we consider that each belief base is composed of a number of statements and the $n$ necessary statements with $\omega_i \geq 0$ with credence 1. We also restrict our class of hypotheses $\mathcal{F}$ to the *dual space* $\Omega^*_{\geq}$, so that each formula $\phi^{(j)}$ is associated to a $n$-dimensional vector $(\phi_1^{(j)}, \ldots, \phi_n^{(j)}) \in \mathbb{R}^n$ and to a linear inequality of the form given in Equation (7).

$$\sum_{i=1}^{n} \phi_i^{(j)} \omega_i \geq 0 \tag{7}$$

With $\mathcal{F}$ and $\Omega$ specified, it is possible to give computational results for the various forms of possibilistic inference semantics.

## 3.2 Computational results

According to Definition 1, inferring $\phi$ on a consistent possibilistic base is made by checking that the set of totally possible worlds is included in $[\![\phi]\!]$. This is equivalent to proving that no totally possible $\omega$ is in $\Omega \setminus [\![\phi]\!]$.
When $\Omega$ is a set of propositional variables and $\mathcal{F}$ the propositional formulae over them, checking whether a possibilistic base is consistent or not is DP-complete, and inference is computationally difficult [Lang, 2000]. With $\Omega = \mathbb{R}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, this check can be performed in polynomial time with linear programming, yielding low complexity results for the various inference problems.

**Proposition 1.** When $\Omega = {\mathbb{R}_+}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, checking whether a given possibilistic belief base is consistent is polytime.

*Proof.* Consider the linear program with decision variables in $\Omega$ consisting in maximizing $\phi^N := \sum_{i=1}^n \omega_i$ subject to all formulae in the base and the fundamental constraints $\omega_i \geq 0$. This linear program is always feasible (because the null vector satisfies all constraints), can be solved in polynomial time [Khachiyan, 1979, Karmarkar, 1984], and the deduction is valid iff the optimum is strictly positive. □

As a corollary, finding the inconsistency level of a belief base is polytime.

**Corollary 1.** When $\Omega = {\mathbb{R}_+}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, given a possibilistic belief base $\Gamma$, computing $Inc(\Gamma)$ is polytime.

*Proof.* Finding the inconsistency level can be performed with a binary search on $\alpha$, by checking whether $\Gamma_{\geq\alpha}$ is consistent or not, as proposed in [Lang, 2000]. It requires $O(\log_2 |\Gamma|)$ calls to a polynomial algorithm, hence it is still polytime.

□

For our three semantics of interest, inference over a consistent belief base is polytime.

**Corollary 2.** When $\Omega = \mathbb{R_+}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, given a possibilistic belief base $\Gamma$, a formula $\phi$ and a level $\alpha \in [0,1]$, if $\Gamma_{\geq\alpha}$ is consistent, then deciding whether $\Gamma \models_X (\phi, \alpha)$ is polytime whatever $X \in \{pi, nt, s\}$.

*Proof.* For all three semantics, inferring $(\phi, \alpha)$ amounts to checking whether $\Gamma_{\geq\alpha} \cup (-\phi, \alpha)$ is inconsistent. For $\models_{pi}$, it is direct via Corollary 1. As the base is consistent, it is also the case for $\models_{nt}$ and $\models_s$ because all subbases are consistent. □

**Corollary 3.** When $\Omega = \mathbb{R_+}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, given a possibilistic belief base $\Gamma$, a formula $\phi$ and a level $\alpha \in [0,1]$, deciding whether $\Gamma \models_{nt} (\phi, \alpha)$ is polytime.

**Proof 1.** *Consistent case in Corollary 2. For the inconsistent case, computing $Inc(\Gamma)$ is polytime. As $\Gamma_{>Inc(\Gamma)}$ is always consistent, inferring with $\models_{nt}$ on an inconsistent base is polytime.*

When considering *safe* possibilistic inference, the requirement to find a consistent subbase makes the problem more computationally demanding. Fortunately, Proposition 2 ensures that it remains in NP.

**Proposition 2.** When $\Omega = \mathbb{R}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, given a possibilistic belief base $\Gamma$, a formula $\phi$ and a level $\alpha \in [0,1]$, deciding whether $\Gamma \models_s (\phi, \alpha)$ is NP-complete.

*Proof. Membership :* given a subset $\Gamma^*$ of the belief base, checking whether $\Gamma^*$ is consistent is polytime. If it is, then checking whether $\Gamma^* \models_{pi} (\phi, \alpha)$ is polytime.
*Hardness :* reduction from VERTEX COVER [Karp, 1972]. From an instance $\langle V, E, K\rangle$ of VERTEX COVER, we build an instance $\langle \Gamma, (\phi, \alpha)\rangle$ of SAFE POSSIBILISTIC INFERENCE as follows. The parameter set is $\Omega := (R_+)^{E \cup V \cup \{\alpha,\beta\}}$. For each edge $e$ and vertex $u \in e$, we denote $\phi^u_e$ the formula $\omega_e + \omega_\alpha \leq \omega_u$, and $\Gamma$ the possibilistic belief base containing all these formulae with credence 1. Let $\phi^\#$ the formula $\sum_{v \in V} \omega_v + \omega_\beta \leq K\omega_\alpha$, and $\phi : \sum_{e \in E} \omega_e + \omega_\beta \leq K\omega_\alpha$. We claim $\Gamma, (\phi^\#, 1) \models_t (\phi, 1)$ iff there is a vertex cover of $(V, E)$ of cardinality $\leq K$.
Indeed, suppose $U \subseteq V$ covers $E$ with $|U| \leq K$, and consider the sub-base $\Gamma_U$ containing all formulae $\phi^u_e$ for $e \in E$ and $u \in e \cap U$, with credence 1. First, the subbase $\Gamma_U \cup \{(\phi^\#, 1)\}$ is consistent, because the interpretation where $\omega_e = 0$ for all edges in $E$, $\omega_u = \omega_\alpha$ for all vertices in $U$, $\omega_v = 0$ for all vertices in $V \setminus U$ and $\omega_\beta = 0$ satisfies all its formulae. Second, it allows to deduce $\phi$ with credence 1, because, for every interpretation of $\omega \in \Omega$ satisfying both $\Gamma_U$ and $\phi^\#$, we have, by summation of all comparisons in $\Gamma_U$ that $\sum_{e \in E} \omega_e \leq \sum_{u \in U}(\omega_u - \omega_\alpha) = \sum_{u \in U} \omega_u - |U|\omega_\alpha$. Moreover $\sum_{u \in U} \omega_u \leq \sum_{v \in V} \omega_v$ because all $\omega_u$ are non-negative and $U \subseteq V$. Thus, $\phi^\#$ yields $\sum_{e \in E} \omega_e + \omega_\beta \leq \sum_{v \in V} \omega_v + \omega_\beta \leq K\omega_\alpha$ and $\phi$ holds in every possible world.
Reciprocally, suppose there is a sub-base $\Gamma^* \subset \Gamma \cup \{(\phi^\#, 1)\}$ which is consistent and allows to derive $\phi$. Suppose $\phi^\# \notin \Gamma^*$ : there is no constraint bounding $\omega_\beta$ from above, and starting from any feasible model and letting $\omega_\beta \to +\infty$ yields a feasible model that does not satisfy $\phi$ at some point : a contradiction. For each edge $e \in E$, if $\Gamma^*$ contained no formula $\phi^u_e$ for some $u_e \in e$, there would be no constraint bounding $\omega_e$ from above, leading to a similar contradiction. Thus, the set $U := \bigcup_{e \in E}\{u_e\}$ covers $E$, and because $\phi^\#$ holds, it cannot have cardinality above $K$. □

At this point, we have not given any general results concerning the baseline semantics of possibilistic inference. Observe this semantics still holds if the base is inconsistent, allowing to infer *any* formula (and also its negation) at some level $\alpha > 0$. Computationally, this is a much more difficult problem.

**Proposition 3.** When $\Omega = \mathbb{R_+}^n \setminus \{0\}$ and $\mathcal{F} = \Omega^*_{\geq}$, given a possibilistic belief base $\Gamma$, a formula $\phi$ and a level $\alpha \in [0,1]$, deciding whether $\Gamma \models_{pi} (\phi, \alpha)$ is NP-hard.

*Sketch of proof.* Adaptation of the reduction from VERTEX COVER put forward in the proof of Proposition 2. Let $e^* = \{u^*, v^*\}$ an arbitrary edge in $E$, and consider a slightly modified belief base $\Gamma'$, where the formulae $\phi^{u^*}_{e^*}$ and $\phi^{v^*}_{e^*}$ now have a slightly lower credence 0.9 instead of 1. The formula $\phi$ can be inferred with $\models_{pi}$ at level 0.9 iff there is a vertex cover, and at level 1 iff there is a vertex cover without $e^*$. □

The question of membership in NP is left open.

### 3.3 Certified inference

The last section proposed a calculus for possibilistic logic based on linear programming. This is a satisfying solution from the computational point of view, allowing tractable inference, even though NP-completeness is hardly scalable, it is sufficient to process small instances. Nevertheless, linear programming is certainly not human-friendly, and relying on a solver to check the validity of inference does not seem to fulfil the requirement of transparency for trustworthy AI. Thus, we propose to support inference with evidence allowing to check its adequacy. The main tool in this endeavour is Farkas'lemma : a system of linear inequalities over $\mathbb{R}^n$ $\phi^1 \geq 0, \ldots, \phi^m \geq 0$ entails $\phi \geq 0$ if, and only if, $\phi$ is a convex combination of the $\phi^k$. Thus, deduction made under the assumptions of Corollary 2 can be supported with a certificate proving its soundness.

**Reasoning w.r.t. safe inference.** We propose a certificate for safe inference.

**Definition 4.** When $\Omega = \mathbb{R}^n \setminus \{0\}$, a primal/dual (or p/d)-certificate is an ordered pair $(\omega^*, \lambda^*)$ where $\omega^* \in \Omega$ and $\lambda^*$ is a tuple of non-negative numbers. Given a possibilistic belief base $\Gamma = \{(\phi^{(1)}, \alpha_1), \ldots, (\phi^{(m)}, \alpha_m)\}$, a formula $\phi$ and a level $\alpha \in [0,1]$, we write $\Gamma \vdash_s^{(\omega^*, \lambda^*)} (\phi, \alpha)$ when all following conditions are satisfied :

i) the length of $\lambda^*$ is equal to $m$, the cardinality of $\Gamma$ ;
ii) for all $1 \leq k \leq m$, if $\lambda^*_k > 0$ then $\phi^{(k)}(\omega^*) \geq 0$ ;
iii) for all $1 \leq k \leq m$, if $\lambda^*_k > 0$ then $\alpha_k \geq \alpha$ ; and

iv) $\phi \geq \sum_{k=1}^{m} \lambda_k^* \phi^{(k)}$.

We write $\Gamma \vdash_s (\phi, \alpha)$ when there is a p/d-certificate $(\omega^*, \lambda^*)$ such that $\Gamma \vdash_s^{(\omega^*,\lambda^*)} (\phi, \alpha)$.

**Proposition 4.** Syntactic deduction $\vdash_s$ is sound and complete w.r.t. $\models_s$.

*Proof.* Define $\Gamma^* := \{(\phi^{(k)}, \alpha_k) : \lambda_k^* > 0\}$. Condition ii) ensures the consistency of $\Gamma^*$, as witnessed by the totally possible $\omega^*$. Condition iii) ensures the credence level of $\Gamma^*$ is at least $\alpha$, warranting inference at this level. Condition iv) ensures the conclusion $\phi$ is in the convex span of the formulae in $\Gamma^*$. All these conditions are necessary for safe possibilistic inference, and together they are sufficient. □

**Reasoning w.r.t. non-trivial inference.** There are two ways to perform non-trivial inference :

— either to compute the inconsistency level $Inc(\Gamma)$ beforehand ; maybe certify it with a primal/dual certificate $(\omega^*, \lambda^*)$ such that $\sum_k \lambda^* \phi^k = -\sum_k \omega_k$ (thus $\Gamma_{>Inc(\Gamma)}$ is consistent) and for all $1 \leq k \leq m$, if $\alpha_k > Inc(\Gamma)$ then $\phi^{(k)}(\omega^*) \geq 0$, and if $\alpha_k \leq Inc(\Gamma)$ then $\lambda_k = 0$ (thus $\Gamma_{\geq Inc(\Gamma)}$ is inconsistent) ; then perform inference with the consistent base (maybe supporting it with a dual certificate).

— or to perform safe inference restricted to a stratified consistent subbase $\Gamma^* \subseteq \Gamma_{>Inc(\Gamma)}$.

**Definition 5.** Under the same assumption as Definition 4, we write $\Gamma \vdash_{nt}^{(\omega^*,\lambda^*)} (\phi, \alpha)$ when conditions i), ii), iii) and iv) are satisfied, as well as :

v) for all $1 \leq k \leq m$, if $\alpha^k \geq \alpha > Inc(\Gamma)$ then $\phi^{(k)}(\omega^*) \geq 0$.

We write $\Gamma \vdash_{nt} (\phi, \alpha)$ when there is a p/d-certificate $(\omega^*, \lambda^*)$ such that $\Gamma \vdash_{nt}^{(\omega^*,\lambda^*)} (\phi, \alpha)$.

**Proposition 5.** Syntactic deduction $\vdash_{nt}$ is sound and complete w.r.t. $\models_{nt}$.

*Proof.* This is corollary of the fact non-trivial inference is simply safe inference restricted to the case where the subbase $\Gamma^* \subseteq \Gamma_{>Inc(\Gamma)}$. Condition v) enforces this. □

As a direct consequence from Propositions 4 and 5, $\vdash_s$ and $\vdash_{nt}$ are not harder than their respective semantics, allowing NP-complete and polynomial-time inference.

**Reasoning with possibilistic inference .** Defining a concise certificate for possibilistic inference remains an open question, conjectured to be equivalent to asserting whether deciding $\models_{pi}$ belongs to NP.

### 3.4 Argued consequence and syntaxes

$\models_s$ originates from [Benferhat et al., 1999]. In the paper, a syntax for safe inference is defined. It differs from Definition 4 as it was not specifically built for linear numerical bases. However, even though the syntaxes are different, their properties are similar.

The generic definition of an argued consequence is given in Definition 6. A subbase $\Gamma^*$ is an argued consequence for a formula $\phi$ if it respects the following properties. We note $\vdash_{ac}$ this syntax.

**Definition 6.** **([Benferhat et al., 1999], Argued consequence)**

$$\begin{cases} 1. & Inc(\Gamma^*) = 0, \\ 2. & \Gamma^* \vdash_{pi} \phi \text{ (relevance)}, \\ 3. & \forall \phi_j \in \Gamma^*, \Gamma^* - \{\phi_j\} \nvdash \phi \quad \text{(economy)}, \end{cases}$$

The relevance and consistency properties are also contained in $\vdash_s$, as in Definition 4. Relevance as Farkas' criterion of infeasibility (through the decomposition), and consistency as a consistent subbase is searched. However, the minimality property is implicit in $\vdash_s$. We have no guarantee that our certificate will be minimal. Therefore, it is not possible to enounce that the two syntaxes are exactly equivalent, but they share two strong properties, the relevance and the consistency of the certificate.

Hence, we have defined three different semantics and syntaxes. One general for the consistent case and two allowing to make non-trivial and safe decision from an inconsistent base. They all have interesting computational results and were linked with Farkas' Lemma, in order to be able to provide a certificate to our user. This point will be developed in the next section, accompanied by illustrations of notions that have been defined in the two last sections. It will be presented through the MCDA example.

## 4 A possible application : MCDA

### 4.1 MCDA in a nutshell

In Multi-Criteria Decision Aiding, our aim is to aid our user (DM) to choose between several alternatives evaluated on several criteria. For example, she might have to choose a hotel in Dijon and her criteria are the price, the number of stars, the distance to the conference center, and the presence/absence of breakfast. We want to learn her preferences. A DM might prefer a cheap hotel while another DM might want an expensive one close to the commute. The common way to model preferences is to consider a weight vector $\omega$ defined over $n$ continuous finite domains, $n$ being the number of criteria. Numerous different models exist, from the weighted sum, to Choquet's integral and the OWA/WOWA. For an extensive review, refer to [Bourdache, 2020]. To conduct the elicitation process, we ask her questions in the form of linear inequalities.

**Example 4.** *Let us take an example where we have to help the DM choose between several alternatives, each described by three criteria. For each criterion, the higher the score the better. The alternatives are presented in Table 5. For illustration purpose, it will be considered that the DM's preferences can be modelled through a weighted sum, one of the simplest numerical models. It is parameterized by a vector $\omega \in \Omega = (\mathbb{R}_+)^3$. It represents the preferences of the DM as follows : given two alternatives $x, y$, $x$ is preferred to $y$, denoted $x \succsim_\omega y$ iff*

| Alternative | Criterion 1 | Criterion 2 | Criterion 3 |
|---|---|---|---|
| $a$ | 10 | 2 | 6 |
| $b$ | 8 | 3 | 7 |
| $c$ | 6 | 6 | 4 |
| $d$ | 4 | 9 | 5 |

TABLE 5 – Alternatives comparison

$$\sum_{i=1}^{3} x_i\omega_i \geq \sum_{i=1}^{3} y_i\omega_i \tag{8}$$

*Given some preference information under the form of a consistent belief base $\Gamma$, it is customary to write $x \succsim_\Gamma y$ if Equation (8) holds for all $\omega$ compatible with all beliefs. We shall keep this notation, but we assume each comparative statement of $\Gamma$ comes with an assessment of its credence, and in turn we provide an indication of credence for inferred beliefs.*
*The DM states she prefers $c$ to $a$, a belief represented by $6\omega_1 + 6\omega_2 + 4\omega_3 \geq 10\omega_1 + 2\omega_2 + 6\omega_3$ or, more succinctly, the formula $-4\omega_1 + 4\omega_2 - 2\omega_3 \geq 0 \in \Omega^*_{\geq}$. Besides comparative statements, more general beliefs can be expressed with linear comparisons. For instance, given criteria are expressed on the same scale, "criterion 2 is more important than criterion 1" is represented by $\omega_2 \geq \omega_1$.*

We perform an active and incremental elicitation process. Examples from the literature can be found in [White et al., 1984, Greco et al., 2008, Jacquet-Lagreze and Siskos, 1982, Adam and Destercke, 2024]. Our aim is to fit the preferences of the DM by restricting the parameters of our numerical model. When the space of totally possible $\omega$ is small enough, we are able to infer on the base by using the syntaxes we defined in the last section.

## 4.2 MCDA and logic of linear comparison

One can see that the MCDA framework we just defined is very close to our logic of linear inequalities. However, several points need to be clarified.

Firstly, very few works in the field of MCDA have tried to associate a credence degree with each piece of information we use to infer. This idea is evoked at the end of [Greco et al., 2008] but, to our knowledge, the only paper that has tried to develop this idea is [Adam and Destercke, 2024]. This is due to a central question around this work; is it relevant to ask our DM for $\alpha_j$ ? This question has already been addressed in Section 2. In the MCDA setup, it is also a maximum purchase price for a ticket for an uncertain lottery. The limitation of current robust elicitation frameworks, as defined in [Greco et al., 2008], is that they assume certain knowledge. We believe that this representation is too naive. This is why we advocate that a rational agent would rather choose a maximum purchase price below the winning prize. We extend the robust approach by allowing the agent to express her doubts on the information she gives us, instead of blindly believing all that she says.

The rest of this section will consist of an example using the tools we have defined in the 2 last sections.

## 4.3 An example to illustrate the developed tools

We continue Example 4. Figure 1 represents $\Omega$ [2].

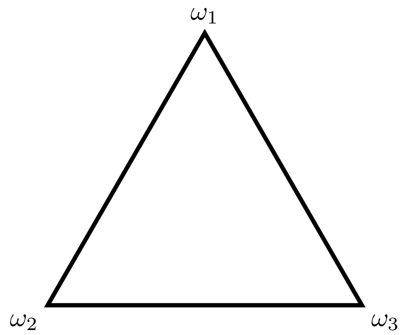


FIGURE 1 – Barycentric representation of the set of worlds $\Omega$.

In this representation, $\Gamma^0 = \{(\omega_1 \geq 0, 1), (\omega_2 \geq 0, 1), (\omega_3 \geq 0, 1)\}$. For a reminder, these are the necessary statements on each $\omega_i$. She gives us a preference :

$$(\phi^{(1)} := \omega_1 \geq \omega_2, \alpha_1 := 0.9)$$

In other words, she thinks that criterion 1 is more important than criterion 2 with a certainty of 0.9. We have $\Gamma^1 = \{(\omega_1 \geq 0, 1), (\omega_2 \geq 0, 1), (\omega_3 \geq 0, 1), (\omega_1 \geq \omega_2, 0.9)\}$. According to our interpretation of such a pair and Equation (1), Figure 2 represents the updated space of parameters.

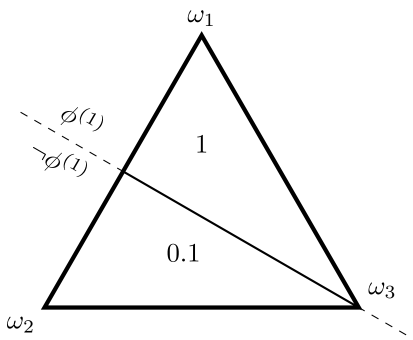


FIGURE 2 – The possibility distribution $\pi_{\Gamma^1}$.

The upper half-space ($[\![\phi^{(1)}]\!]$) is constituted of all the $\omega$ such that $\omega_1 \geq \omega_2$, therefore with a $\pi_\Gamma(\omega) = 1$. The lower one is constituted of all the $\omega$ not satisfying $\phi^{(1)}$, hence with a possibility of $1 - 0.9$. She adds a second statement.

$$(\phi^{(2)} := \omega_3 \geq \omega_1, \alpha_2 := 0.8)$$

2. Without loss of generality, a normalization constraint ($\omega_1 + \omega_2 + \omega_3 = 1$) is added to permit the use of barycentric coordinates for an easier representation

Hence, $\Gamma^2 = \{(\omega_1 \geq 0, 1), (\omega_2 \geq 0, 1), (\omega_3 \geq 0, 1), (\omega_1 \geq \omega_2, 0.9), (\omega_3 \geq \omega_1, 0.8)\}$, we obtain the possibility distribution over $\Omega$ shown in Figure 3. Due to the minimum in Equation (2), for the left part of the triangle (i.e. $\omega$ not satisfying any of the two constraints), we take the minimum value of possibility.

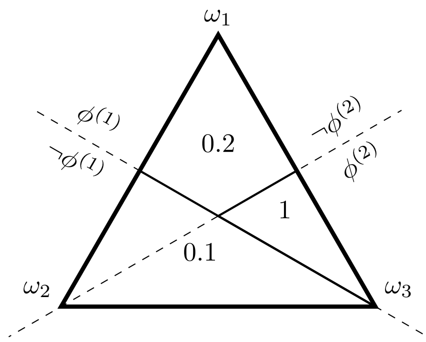


FIGURE 3 – The possibility distribution $\pi_{\Gamma^2}$.

$\Gamma^2$ is consistent, as witnessed e.g. by the primal certificate $\omega = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. To verify whether $a \succeq_{\Gamma^2} d$ can be inferred, $\phi^{a \succ d} := (6\omega_1 - 7\omega_2 + \omega_3 \geq 0)$ must hold for all totally possible $\omega$. First, the proof by refutation is drawn in Figure 4. The half-space corresponding to the negation of $\phi^{a \succ d}$, with a certainty of 1, is added to $\Gamma^2$.

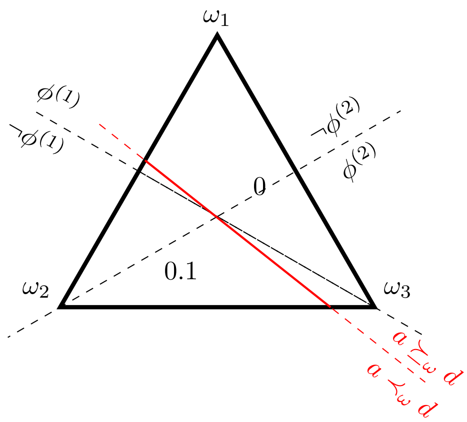


FIGURE 4 – The possibility distribution induced by $\Gamma^2 \cup (\neg\phi^{a \succ d}, 1)$.

The distribution in Figure 4 is inconsistent as no $\omega$ belongs to all half-spaces. This semantically proves that our DM prefers alternative $a$ over alternative $d$, with a certainty of 0.8, because the maximum of $\pi_{\Gamma^2}$ over the worlds where $d \succ a$ is 0.2. We establish this preference syntactically by providing a dual certificate. The inequality system is presented in Equation (9).

$$\Gamma^2 = \begin{cases} \omega_1 \geq 0 & (d^{(1)}1, 1) \\ \omega_2 \geq 0 & (d^{(2)}, 1) \\ \omega_3 \geq 0 & (d^{(3)}, 1) \\ \omega_1 - \omega_2 \geq 0 & (\phi^{(1)}, 0.9) \\ \omega_3 - \omega_1 \geq 0 & (\phi^{(2)}, 0.8) \end{cases} \tag{9}$$

Observe $\phi = 7\phi^{(1)} + \phi^{(2)}$, which allows to deduce $\phi^{a \succ d}$ from $\Gamma^2$ with certainty $min_{i \text{ s.t}\lambda_i \neq 0} \alpha_j = 0.8$.
Let us add a new $\phi^{(3)}$ to our base that breaks consistency.

$$(\phi^{(3)} = c \succeq_\omega b, \alpha_3 := 0.7)$$
$$\text{with } \phi^{(3)} \Leftrightarrow -2\omega_1 + 3\omega_2 - 3\omega_3 \geq 0$$

We now consider $\Gamma_3 := \Gamma_2 \cup \{(\phi^{(3)}, 0.7\}$.
The possibility distribution induced by $\Gamma^3$ is displayed on Figure 5.

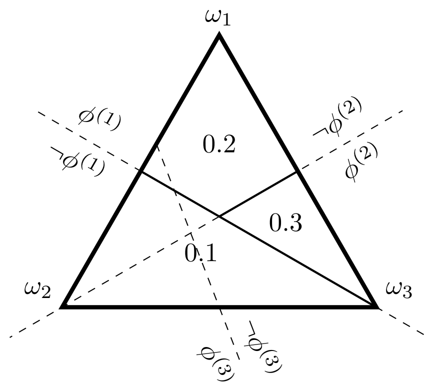


FIGURE 5 – The possibility distribution $\pi_{\Gamma^3}$.

$\Gamma^3$ is inconsistent, no $\omega \neq 0$ is a solution of its system. Observe that the convex combination $7\phi^{(1)} + 4\phi^{(2)} + 2\phi^{(3)}$ ensures $-\omega_1 - \omega_2 - 2\omega_3 \geq 0$, while the non-negativity of $\omega_1, \omega_2, \omega_3$ entails $\omega_1 + \omega_2 + 2\omega_3 \geq 0$. Both cannot be satisfied if $\omega$ is non null. We illustrate how inference based on different semantics produces different outcomes.
*Possibilistic inference* $\models_{pi}$ allows to infer *any* formula $\phi$ from $\Gamma_3$, but with a level $\alpha = 1 - \max_{\neg\phi} \pi_{\Gamma^3} \in \{0.1, 0.2, 0.3\}$. This is a much more nuanced version of the principle of explosion of classical logic which puts all formulae on the same level.
*Non-trivial inference* $\models_{nt}$ prescribes to ignore the less certain stratum at level 0.7 which provokes inconsistency. Thus, from this viewpoint, $\Gamma^3$ is equivalent to its safe strata $\Gamma^2$. Observe that, if beliefs compatible with $\Gamma^2$, but no more certain than 0.7 were present in the base, they would have been "drowned".
*Unsafe inference* could be performed by leveraging dual certificates but ignoring the requirement of a primal certificate ensuring local consistency. Indeed, define $\phi := (a \succeq_\omega d) \Leftrightarrow (6\omega_1 - 7\omega_2 + \omega_3 \geq 0)$ and observe $\phi = 10\phi^{(1)} + 2\phi^{(2)} + \phi^{(3)} + 2\omega_3$, allowing to *unsafely* derive $\phi$ at level $\alpha = 0.7$. However, the basis of this certificate is $\Gamma^* = \Gamma^3$ and is inconsistent, undercutting the ar-

gument supporting $\phi$ : in every world where $\phi^{(1)}$, $\phi^{(1)}$ and $\phi^{(3)}$ hold, $\phi$ holds, but there is none.

*Safe inference* allows to make deductions based on any (maximally) consistent subbase of $\Gamma^3$ : either $\Gamma^2$ by ignoring $\phi^{(3)}$, or the bases obtained by ignoring respectively $\phi(1)$ or $\phi^{(2)}$. This is much more versatile than non-trivial inference, at the cost of solving a NP-complete problem.

In this section, the logic of linear comparison has been illustrated through MCDA. We applied our syntaxes to infer on a belief base in the context of the MCDA problem. However, other problems solved by numerical models could also be used within this framework, such as scheduling. This idea is not recent, such as [Dubois et al., 2003] gave an overview on how possibility theory can be applied to scheduling. This is left for further research, such as the decision procedure.

# 5 Related works

## 5.1 Possibility theory

Quantitative Possibility Theory, as defined in [Dubois and Prade, 1998], is different from what we want to achieve in this work. QPT defines the semantics of another possibility theory, where $\alpha$ is not defined over an ordinal set but over a continuous domain, as in probability theory.

To our knowledge, the only example of work combining possibility theory and numerical preferences (except [Adam and Destercke, 2024]), is [Kaci and Prade, 2008]. The fundamental difference between our approaches is that, instead of considering the priority as a degree of certainty, they choose to consider it as a degree of intensity.

Computational results have not been the main focus in possibilistic logic, except for [Lang, 2000] and papers on combinatorial logic. In this paper, results have been given on numerical linear logic, results absent from the literature.

## 5.2 Numerical models

The proposed extension in [Greco et al., 2008] extends the robust approach. In some papers as [Greco et al., 2008], the hypothesis is that the DM never commits any mistake. Even if we drop this hypothesis, no other work has tried to associate a degree of certainty to each piece of information.

# 6 Conclusion

In this paper, a theoretical framework combining logical frameworks and numerical models has been presented. Our main objective is to benefit from their respective qualities. Numerical models, in our case linear inequalities, are interesting from a computational point of view. A logical framework provides clear semantics and tools to infer, even in case of inconsistency. We chose to use Possibility Theory, as it is a formalism to represent uncertainty that comes with several good properties, as the conservation of the bounds value during the elicitation process thanks to the minimum specificity principle and its non-necessity to normalize the distribution of possibility at every step.

Semantics and syntaxes for inferring over a base built out of formulas and their associated degree of certainty have been defined : a polytime semantics and syntax to infer on a consistent base, another polytime couple to infer on an inconsistent base but with the downside that several pieces of information are not considered, and a last one to infer on an inconsistent base, which is at least NP-hard. All of these syntaxes have been linked to Farkas' criterion of contradiction, as used in the explanation literature. We believe that an analyst would be more confident in the decision returned by the elicitation if each of them was associated with a degree of certainty. Furthermore, Farkas' provides a certificate of infeasibility, which can be seen as an early form of explanation.

We finally gave a possible application to our logic of linear combination, in the world of MCDA. It was necessary to give a justification for the existence of the degree of uncertainty associated with each formula. We believe that they can be seen as a maximum purchase price in a game when the agent wins if the information she gives is proved to be correct. Other examples could have been investigated, such as fuzzy scheduling.

Future works include searching for new syntaxes to infer despite the inconsistency. Another peculiar point of interest is the decision process in case of inconsistency. It may be linkable to bipolar argumentation, as defined in [Amgoud et al., 2008, Al Anaissy, 2024]. Finally, exploring the explanation process for a recommendation coming from a belief base is an important subject.

# Remerciements

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